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Angular analysis of bremsstrahlung in α-decay

S.P. Maydanyuk[a] and V.S. Olkhovsky[b]

Institute for Nuclear Research, National Academy of Sciences of Ukraine, prosp. Nauki, 47, Kiev-28, 03680, Ukraine

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This paper is dedicated to the memory of Dr Ivan Egorovich Kashuba —a brilliant scientist with bright nature who worked in science till his last days.

Abstract. A new quantum electrodynamical method of calculations of bremsstrahlung spectra in the α-decay of heavy nuclei taking into account the angle between the directions of α-particle motion (or its tunneling) and photon emission is presented. The angular bremsstrahlung spectra for ^{210}Po have been obtained for the first time. According to calculations, the bremsstrahlung in the α-decay of this nucleus depends extremely weakly on the angle. Taking into account nuclear forces, such dependence is not changed visibly. An analytical formula of the angular dependence of the bremsstrahlung spectra is proposed and gives its harmonic behavior. The extremal values of the angle, at which the bremsstrahlung has maximal and minimal values, have been found.

PACS. 23.60.+e Alpha decay – 41.60.-m Radiation by moving charges – 23.20.Js Multipole matrix elements – 03.65.Xp Tunneling, traversal time, quantum Zeno dynamics

1 Introduction

Experiments [1–3] with measurements of bremsstrahlung (Br) spectra in the α-decay of the nuclei ^{210}Po, ^{214}Po, ^{226}Ra and ^{244}Cm have caused an increased interest. One of the key ideas of the fulfillment of such experiments consists in finding a method of extraction of a new information about α-decay dynamics from the Br spectra (and a detailed information about the dynamics of tunneling). One can note a certain difference between the Br spectra [1] and [2,3] for ^{210}Po, obtained experimentally for the values 90° and 25° of the angle between the directions of the α-particle propagation and the photon emission (these experiments and the difference of their spectra are discussed in [4,5]). One can explain such difference between the Br angular spectra on the basis of the following idea: *the Br intensity depends on the directions of emission of the photons and motion (with possible tunneling) of the α-particle relatively to the daughter nucleus.* In such a way, a three-dimensional picture of the α-decay with the accompanying Br in the spatial region of nuclear boundaries has been devised.

However, if the Br intensity varies enough visibly with changing the angle value, then one can suppose, that the photon emission is able to influence essentially the α-decay dynamics and, therefore, to change all of its characteristics. From this point of view, the discussions [4,5] open a way for obtaining a new information about the α-decay —through the *angular analysis of the Br during the α-decay.* But for such researches a model describing the Br in the α-decay, which takes into account the value of the angle between the directions of the α-particle propagation (or tunneling) and the photon emission, is needed.

In the theoretical aspects, some progress has been made here. One can note models of calculations of the Br spectra in the α-decay, developed on the basis of quantum electrodynamics with use of perturbation theory: the first paper [6] where a general quantum-mechanical formalism of the calculation of the Br spectra in the α-decay is proposed and the Br spectrum for ^{210}Po inside the photons energy region up to 200 keV was estimated (even until the fulfillment of the first experiments); essentially improved models in the dipole approximation [7,8] and in the multipolar expansion [9] of photons current (wave function) with application of the *Fermi golden rule*; an approach [10] of the calculation of the Br spectra with realistic barriers of the α-decay, models [11–13,8] developed in semiclassical approximation (see also the Br spectra calculations in [3]), *instant accelerated models* [1, 9] constructed on the basis of classical electrodynam-

[a] e-mail: maidan@kinr.kiev.ua
[b] e-mail: olkhovsk@kinr.kiev.ua

ics (see also [12]), methods [13–15,11,12], directed on a non-stationary description of the α-decay with the accompanying Br and the calculations of such non-stationary characteristics as *tunneling time*. One can recall also papers [16–19] with study of dynamics of subbarrier tunneling in the α-decay; an effect, first described in [20] and named *Münchhausen effect*, which increases the barrier penetrability due to charged-particle emission during its tunneling and which can be extremely interesting for further study of the photon bremsstrahlung during subbarier tunneling in the α-decay. However, one needs to say that at this stage the calculations of the Br spectra by all these approaches are reduced to obtaining their integral (or averaged by angles) values and, therefore, they do not allow to fulfill an angular analysis of the experimental Br spectra (here, one can quote an approach in [14] based on classical electrodynamics, which shows a way for obtaining the angular spectra (see (15) and (17), p. 999), however here we shall use the direct quantum-mechanical approach of the Br spectra calculation, which describes the quantum effect of the *subbarrier Br* more precisely).

In [21] we had developed a multipolar method which takes into account the angle between the directions of the α-particle propagation and the photon emission. But the angular integrals used in this method are difficult enough to be obtained and some approximations are used, there was a convergence problem with the calculations of multipoles of larger orders, only the angular dependences of the matrix elements of the selected multipoles $E1$ and $M1$ were found, while it is interesting to know also the angular dependence of the total Br spectra. Moreover, computer calculations of the Br spectra and their angular analysis will be essentially more complicated, if one passes from the type of the potential used in this paper (and also in [7,9]) to realistic potentials. In this sense, the approach proposed in [21] is not convenient enough.

In this paper we present a new method for the angular calculations of the Br spectra in the α-decay of nuclei (started in [22], with the resolution of a convergence problem in the Br spectra calculations existing in [22]). In our approach we introduce a simplified transformation, which reduces the complicated angular formalism of the calculation of the Br spectra (presented in [21]) to a maximally simple form (this makes the method clearer and more *comprehensible*), with keeping the calculating accuracy as good as possible, where only one angle from all angular parameters is used —the angle used in experiments [4,5] for ^{210}Po. We show that this proposed transformation works in the low-energy region of photons and, therefore, it can be applied to the analysis of all up to now existing experimental data of Br spectra in the α-decay of spherical nuclei (we relate ^{210}Po to them). In this paper we present the results of the calculations of the angular spectra of the Br in the α-decay for ^{210}Po (the Br angular spectra in the α-decay have been obtained for the first time). We show how the Br spectra are changed after deformation of the form of the α-decay barrier as a result of the correction of a component of the α-nucleus potential of nuclear forces (we did not find such calculations in other papers). An analysis of the problem of the calculations convergence of the Br spectra in the α-decay is included in the paper, whose resolution plays a key role for obtaining of the reliable values of the spectra.

2 A formalism of the calculations of the bremsstrahlung spectra in the stationary approach

We shall consider the decay of the nucleus as a decay of the compound quantum system: α-particle and daughter nucleus. The α-particle is the electrically charged particle and during its motion inside the electromagnetic field of the daughter nucleus it emits photons. The spontaneous emission of the photon changes the state of the compound system, which is described by its wave function. For a quantitative estimation of the Br of photons we use a transition of the system from its state before the photon emission (we name such state the *initial i-state*) into its state after the photon emission (we name such state the *final f-state*). One can define a matrix element of such transition of the system and on its basis find the Br probability during the α-decay (for convenience, we denote it as $W(w)$). According to [21], we obtain

$$W(w) = N_0 k_f w \big|p(w)\big|^2, \quad N_0 = \frac{Z_{\text{eff}}^2 e^2}{(2\pi)^4 m},$$
$$k_{i,f} = \sqrt{2mE_{i,f}}, \quad w = E_i - E_f, \tag{1}$$

where $p(w)$ has the form

$$p(w) = \sum_{\alpha=1,2} \mathbf{e}^{(\alpha)*} \int_0^{+\infty} \mathrm{d}r \int r^2 \psi_f^*(\mathbf{r}) e^{-i\mathbf{kr}} \frac{\partial}{\partial \mathbf{r}} \psi_i(\mathbf{r}) \mathrm{d}\Omega. \tag{2}$$

Here Z_{eff} and m are the effective charge and the reduced mass of the system, $E_{i,f}$, $k_{i,f}$ and $\psi_{i,f}(\mathbf{r})$ are the total energy, wave vector and wave function of the system in the initial i-state or in the final f-state (according to the index i or f in use), $\psi_i(\mathbf{r})$ and $\psi_f(\mathbf{r})$ are the wave function of the system in the initial i- and the final f-states, $\mathbf{e}^{(\alpha)}$ is the polarization vector of the photon emitted, $\mathbf{k}$ is the photon wave vector, $w = k = \big|\mathbf{k}\big|$ is the photon frequency (energy). The vector $\mathbf{e}^{(\alpha)}$ is perpendicular to $\mathbf{k}$ in the Coulomb calibration. We use the following system of units: $\hbar = 1$ and $c = 1$. Notations used are in accordance with [21]. Similar expressions for the Br probability are used in [7–9] with further application of the Fermi golden rule.

In accordance with the main statements of quantum mechanics, the wave functions of the system in the states before and after the photon emission are defined inside all the space region, including the region of the subbarrier tunneling. A definition of the matrix element of the transition of this system requires the consideration of all the space region of the definition of the wave functions of this system in the two states. Therefore, *we should include the tunneling region into the definition of the matrix element*

of the Br, irrespective of whether we know that the photons emission is possible during tunneling or not.

Let us consider a subintegral expression in (2). Here, the wave function $\psi_i(\mathbf{r})$ for the initial i-state and the wave function $\psi_f(\mathbf{r})$ for the final f-state take into account the directions of propagation (or tunneling) of the α-particle before the photon emission and after it, respectively; the photons wave function (its main part consists in the exponent $\exp(-i\mathbf{kr})$) points to the direction of propagation of the emitted photon. We see that the quantum-mechanical approach for the calculation of the Br spectra initially gives a detailed angular information about the process of α-decay with the accompanying Br.

However, we see that a further development of the approach for the calculations of the Br spectra in the α-decay on the basis of formulas (1) and (2) by other authors (which consists in the calculations of $p(w)$) gives rise to angular averaging of the spectra. And the necessity aroused of constructing an approach, which allows to calculate the Br spectra simply enough (with a possible solution of the convergence problem in computer calculations) by taking into account the angle between the directions of the α-particle propagation (with possible tunneling) and the photon emission and without (essentially) decreasing the accuracy.

3 A simplified angular method of the matrix element calculation

In [22] an approach for the calculation of the Br spectra, allowing to find the dependence of the total Br spectra on the angle between the directions of the α-particle propagation (or tunneling) and the photon emission, was proposed. However, further research has shown that it is extremely difficult to achieve a convergence in the computer calculations of the Br spectra by such approach and, therefore, such a method requires a special development. Here, we propose a subsequent statement of this approach with a resolution of the convergence problem.

Let us rewrite the polarization vectors $\mathbf{e}^{\alpha}$ through the vectors $\boldsymbol{\xi}_{-1}$ and $\boldsymbol{\xi}_{+1}$ of circular polarization with opposite directions of rotation (see [23], p. 42):

$$\boldsymbol{\xi}_{-1} = \frac{1}{\sqrt{2}}(\mathbf{e}^1 - i\mathbf{e}^2), \quad \boldsymbol{\xi}_{+1} = -\frac{1}{\sqrt{2}}(\mathbf{e}^1 + i\mathbf{e}^2). \tag{3}$$

Substituting these values into (2), we obtain

$$p(w) = \sum_{\mu=-1,1} h_{\mu}\boldsymbol{\xi}^*_{\mu} \int\limits_0^{+\infty} \mathrm{d}r \int r^2 \psi^*_f(\mathbf{r}) e^{-i\mathbf{kr}} \frac{\partial}{\partial \mathbf{r}} \psi_i(\mathbf{r}) \mathrm{d}\Omega, \tag{4}$$

where

$$h_{-1} = \frac{1}{\sqrt{2}}(1-i), \qquad h_1 = -\frac{1}{\sqrt{2}}(1+i),$$
$$h_{-1} + h_1 = -i\sqrt{2}. \tag{5}$$

Using the following properties (see [23] p. 44–46, [21]):

$$\frac{\partial}{\partial \mathbf{r}}\psi_i(\mathbf{r}) = -\frac{\mathrm{d}\psi_i(r)}{\mathrm{d}r}\mathbf{T}_{01,0}(\mathbf{n}^i_r),$$
$$\mathbf{T}_{01,0}(\mathbf{n}^i_r) = \sum_{\mu=-1,1}(110|-\mu\mu 0)Y_{1,-\mu}(\mathbf{n}^i_r)\boldsymbol{\xi}_{\mu}, \tag{6}$$
$$(110|1,-1,0) = (110|-1,1,0) = \sqrt{\frac{1}{3}},$$

where $(110|-\mu\mu 0)$ are Clebsch-Gordan coefficients and $\mathbf{T}_{ll',\mu}(\mathbf{n})$ are vector spherical harmonics (see [23], p. 45 and we use quantum numbers $l = m = 0$ in the initial i-state), $Y_{1,\mu}(\mathbf{n}^{i,f}_r)$ are normalized spherical functions (see [24], p. 118–121 (28.7), p. 752–755), we obtain

$$\frac{\partial}{\partial \mathbf{r}}\psi_i(\mathbf{r}) = -\frac{\mathrm{d}\psi_i(r)}{\mathrm{d}r}\sqrt{\frac{1}{3}}\sum_{\mu'=-1,1} Y_{1,-\mu'}(\mathbf{n}^i_r)\boldsymbol{\xi}_{\mu'}. \tag{7}$$

Taking into account (4), (7) and the ortogonality condition of the polarization vectors $\boldsymbol{\xi}^*_{\pm 1}$ and $\boldsymbol{\xi}_{\mp 1}$, we find

$$p(w) = -\sqrt{\frac{1}{3}}\sum_{\mu=-1,1} h_{\mu} \int\limits_0^{+\infty} \mathrm{d}r r^2 \psi^*_f(r)\frac{\partial \psi_i(r)}{\partial r}$$
$$\times \int Y^*_{l'm'}(\mathbf{n}^f_r) Y_{1,-\mu}(\mathbf{n}^i_r) e^{-i\mathbf{kr}} \mathrm{d}\Omega, \tag{8}$$

where $\psi_f(\mathbf{r}) = \psi_f(r)Y_{l',m'}(\mathbf{n}^f_r)$.

Let us consider the vectors $\mathbf{k}$ and $\mathbf{r}$. The vector $\mathbf{k}$ is an impulse of the photon, pointing out its direction of propagation. The vector $\mathbf{r}$ is a radius-vector, pointing out the position of the α-particle relatively to the center of mass of the daughter nucleus and (because the mass of the daughter nucleus is sufficiently larger than the mass of the α-particle) pointing out the direction of its motion (or tunneling). Then the angle between the vectors $\mathbf{k}$ and $\mathbf{r}$ (let us denote it as β) is the angle between the direction $\mathbf{n}_r = \mathbf{r}/r$ of motion (or tunneling) of the α-particle and the direction $\mathbf{n}_{ph} = \mathbf{k}/k$ of the propagation of the emitted photon, *i.e.* it is the angle used in the experiments [4,3, 1]. One can write

$$\exp(-i\mathbf{kr}) = \exp(-ikr\cos\beta), \quad k = |\mathbf{k}|, \quad r = |\mathbf{r}|. \tag{9}$$

Now we make *the following assumptions*:

- the photon emission process does not change the direction of motion (or tunneling) of the α-particle:

$$\mathbf{n}^i_r = \mathbf{n}^f_r, \tag{10}$$

- the angle β is not dependent on the direction of motion of the α-particle outgoing from the nucleus region.

Then, taking into account these assumptions and the ortogonality property of the functions $Y_{lm}(\mathbf{n}_r)$, we obtain the following expression for $p(w,\beta)$:

$$p(w,\beta) = -\sqrt{\frac{1}{3}}\sum_{\mu=-1,1} h_{\mu} \int\limits_0^{+\infty} r^2 \psi^*_f(r)\frac{\partial \psi_i(r)}{\partial r} e^{-ikr\cos\beta}\mathrm{d}r \tag{11}$$

and the following selection rules for the quantum numbers l and m of the final f-state:

$$\begin{array}{ll} \text{the initial state: } l_i = 0, & m_i = 0; \\ \text{the final state: } \quad l_f = 1, & m_f = -\mu = \pm 1. \end{array} \tag{12}$$

4 Spherical wave expansion

For further computer calculations of the integral (11), let us use an expansion of the plane wave in the spherical waves (for example, see [24], p. 144, (34.1)):

$$e^{ikz} = \sum_{l=0}^{+\infty} (-i)^l (2l+1) P_l(\cos\beta) \left(\frac{r}{k}\right)^l \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^l \frac{\sin kr}{kr}, \tag{13}$$

where $z = r\cos\beta$. Introducing the spherical Bessel functions (see [24], p. 139, (33.9), (33.10) and (33.11))

$$j_l(kr) = (-1)^l \left(\frac{r}{k}\right)^l \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^l \frac{\sin kr}{kr}, \tag{14}$$

we obtain

$$e^{-ikr\cos\beta} = \left(e^{ikr\cos\beta}\right)^* = \sum_{l=0}^{+\infty} i^l (-1)^l (2l+1) P_l(\cos\beta) j_l(kr) \tag{15}$$

and from (11) we find

$$p(w,\beta) = -\sqrt{\frac{1}{3}} \sum_{l=0}^{+\infty} i^l (-1)^l (2l+1) P_l(\cos\beta) \times \sum_{\mu=-1,1} h_\mu J_{m_f}(l,w), \tag{16}$$

where

$$J_{m_f}(l,w) = \int_0^{+\infty} r^2 \psi_f^*(r) \frac{\partial \psi_i(r)}{\partial r} j_l(kr)\mathrm{d}r. \tag{17}$$

$J_{m_f}(l,w)$ is a radial integral, independent of the angle β. Now *we obtain an explicit analytical dependence of the matrix element $p(w,\beta)$ on the angle β between the directions of the α-particle propagation and the photon emission* (for the first time, in [21] the angular integrals were obtained for selected components of this matrix element —multipoles $E1$ and $M1$ only, with use of more difficult calculations).

5 The bremsstrahlung in the Coulomb field

Practically, in the numerical calculation of the Br spectra it is convenient to divide the whole region of the integration into two parts: region 1 of a joint action of the Coulomb and nuclear forces not far from the nucleus and region 2, in which one can neglect the action of the nuclear forces in comparison with the action of the Coulomb forces. Our analysis has shown that the attainment of the *convergence* of the Br spectra calculations (which determines their *accuracy*, *reliability* of the found Br spectra) is reached first of all by the correctness of the calculations in the region 2. Namely, in this region one needs to solve the problem of the definition of the external boundary of integration (its increasing leads to an increase of the accuracy of the obtained spectra, but also to an increase of the difficulty of the calculations and analysis), to choose the most effective method of numerical integration (of an improper integral with an oscillating and weakly damping sub-integral function), to solve the problem with the attainment of the needed accuracy and convergence of the calculations. This defines the time necessary for the calculations, minimization of which appears extremely important for the fulfillment of the real analysis of the obtained Br spectra in dependence on the needed parameters. Therefore, maximal simplification of the formulas for the Br spectra in region 2 is in order.

Let us assume that the potential, used in the calculation of wave functions for the initial i- and final f-states in the radial integral (17), in the spatial region of r is of pure *Coulomb* type since the value R_c. We accept R_c as the internal boundary of region 2. One can write the radial integral $J(l,w)$ in (17) in this way

$$J_{m_f}(l,w) = J_{in,m_f}(l,w) + J_c(l,w), \tag{18}$$

where

$$\begin{aligned} J_{in,m_f}(l,w) &= \int_0^{R_c} r^2 \psi_f^*(r,m_f) \frac{\partial \psi_i(r)}{\partial r} j_l(kr)\mathrm{d}r, \\ J_c(l,w) &= \int_{R_c}^{+\infty} r^2 \psi_f^*(r) \frac{\partial \psi_i(r)}{\partial r} j_l(kr)\mathrm{d}r. \end{aligned} \tag{19}$$

The radial integral $J_c(l,w)$ does not depend on the quantum number m of the systems in the final f-state. Then, one can write $p(w,\beta)$ in this way (taking into account (5) for the Coulomb component):

$$p(w,\beta) = p_{in}(w,\beta) + p_c(w,\beta), \tag{20}$$

where

$$\begin{aligned} p_{in}(w,\beta) &= -\sqrt{\frac{1}{3}} \sum_{l=0}^{+\infty} i^l (-1)^l (2l+1) P_l(\cos\beta) \\ &\quad \times \sum_{\mu=-1,1} h_\mu J_{in,m_f}(l,w), \\ p_c(w,\beta) &= \sqrt{\frac{2}{3}} \sum_{l=0}^{+\infty} i^{l+1} (-1)^l (2l+1) P_l(\cos\beta) J_c(l,w). \end{aligned} \tag{21}$$

We see that there is not any interference between the components $p_{in}(w,\beta)$ and $p_c(w,\beta)$ in the calculations of the total value of $p(w,\beta)$, but it exists in calculations of the total Br spectra.

6 The first approximation at l = 0

Legendre's polynomial of the order l equals (for example, see [24], p. 752 (c.1))

$$P_l(\theta) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{(\mathrm{d}\theta)^l}(\theta^2-1)^l, \qquad P_0(\theta)=1, \quad P_1(\theta)=\theta, \quad \theta=\cos\beta. \tag{22}$$

Then at $l=0$, we find

$$p_{in}^{(l=0)}(w,\beta) = -\sqrt{\frac{1}{3}}\sum_{\mu=-1,1} h_\mu J_{in,m_f}(0,w), \qquad p_c^{(l=0)}(w,\beta) = i\sqrt{\frac{2}{3}}J_c(0,w). \tag{23}$$

If for nuclei ^{210}Po, ^{214}Po, ^{226}Ra one uses the potential with parameters as in [21] (and also as in [7,9]), then we find that Br from the internal spatial region up to R_c is extremely small ($p_{in}(w,\beta) \ll p_c(w,\beta)$). According to our estimations, for such potential the Br is 10^{-22}–10^{-24} times smaller than the Br from the external region. Therefore, one can neglect the Br from the internal region, and the total Br can be determined by the Coulomb field inside the barrier region and the external region. From (1) we write down the Br probability in the first approximation at $l=0$:

$$W_{l=0}(w) = N_0 k_f w\left|p_c^{(l=0)}(w,\beta)\right|^2 = \frac{2}{3}N_0 k_f w\left|J_c(0,w)\right|^2. \tag{24}$$

One can conclude that (this has been obtained for the first time):

- The Br probability in the first approximation at $l=0$, formed by the Coulomb field both taking into account nuclear forces of any shape, and without such forces, does not depend on the value of the angle β between the directions of the α-particle propagation (or its tunneling) and the photon emission.
- The Coulomb field is degenerated by the quantum number m. This property distinguishes the Coulomb field from the nuclear forces under consideration in the model. This difference is shown in the matrix elements (23). The nuclear forces participate in the formation of the decay barrier and, therefore, one can consider them approximately as forces working in the spatial region of the barrier, where there is a tunneling. One can assume that one can divide the emissions from the barrier region and from the external region on the basis of the quantum number m_f. It would be interesting to find a possible way of extraction of the Br spectrum from the barrier region (or from the external region) from the experimental Br spectrum on the basis of this property.

7 The second approximation at l = 1

Taking into account (22), we find the Br probability in the second approximation at $l=1$:

$$p_{in}^{(l=1)}(w,\beta) = i\sqrt{3}\cos\beta\sum_{\mu=-1,1} h_\mu J_{in,m_f}(1,w), \qquad p_c^{(l=1)}(w,\beta) = \sqrt{6}\cos\beta J_c(1,w). \tag{25}$$

Neglecting Br from the internal region, we obtain the following expressions for the component of the Br probability of the second approximation at $l=1$:

$$W^{(l=1)}(w,\beta) = N_0 k_f w\left|p_c^{(l=1)}(w,\beta)\right|^2 = 6N_0 k_f w\left|J_c(1,w)\right|^2\cos^2\beta \tag{26}$$

and for the total Br probability in the second approximation at $l=1$:

$$W_{l=1}(w,\beta) = W_{l=0}(w)\left|1-N(w)\cos\beta\right|^2, \qquad N(w) = 3i\frac{J_c(1,w)}{J_c(0,w)}. \tag{27}$$

One can conclude that (this has been found for the first time):

- The dependence of the Br probability in the α-decay on the value of the angle β between the directions of the α-particle propagation (or its tunneling) and the photon emission is of harmonic type (27).
- The consideration of the nuclear forces does not change the dependence of the Br probability in the second approximation at $l=1$ on such angle value.
- Equations (27) allow to analitically determine maxima and minima in the Br spectra as a function of the angle β.

8 Convergence of calculations in the asymptotic region

There is a considerable difficulty in the calculations of the Br spectra for a given nucleus, related to obtaining the radial integrals (19) (or (17)). This difficulty is caused by the fact that such integral is improper, and its sub-integral function is oscillating and damping slowly with increasing r. The function damps weaker with increasing r, the larger region of integration should be taken into account in the numerical integration. For ^{210}Po the damping degree of the sub-integral function is such that for reliable values of the first 2-3 digits for the Br spectrum one needs to take into account (with the highest accuracy of calculation) 1 million oscillations of this function.

As an evident demonstration of this problem, let us consider the one-dimensional integral:

$$\int_a^{+\infty}\frac{\sin x}{x}\mathrm{d}x. \tag{28}$$

An exact analytical value of this integral at $a = 0$, equal to $\pi/2$, is known from the complex variable function theory. The numerical calculation of the integral (with the use of the simple method of trapeziums, the method of Gauss or other methods of numerical integration) allows to quickly obtain the same result also, but with a given degree of accuracy (which determines the region of numerical integration). This proves a convergence of the computer calculations of such integral with a concrete choice of the parameter a. But *at weak increasing of the parameter a the region of numerical integration for obtaining the same calculation accuracy for integral (28) increases remarkably, and, therefore, the difficulty to calculate this integral numerically increases accordingly.* However, the application of the method of the complex variable function theory makes the calculation of such integral simpler again. So, in the example of the simple integral (28) one can meet with the numerical problem of the convergence of the calculations of the improper integrals with the slowly damping, oscillating sub-integral functions.

We perform the analysis of the convergence of the calculation of the integral (19) on the basis of the analysis of the damping of its sub-integral function in the asymptotic region, which is defined by wave functions in the initial i-, final f-states and the by the spherical Bessel function of order l.

For large enough values of r one can use an asymptotic representation of the spherical Bessel function of order l:

$$j_l^{(as)}(kr) = \frac{1}{kr}\sin\left(kr - \frac{\pi l}{2}\right), \tag{29}$$

or

$$\begin{aligned} j_{2n}^{(as)}(kr) &= (-1)^n j_0^{(as)}(kr) = (-1)^n \frac{\sin kr}{kr}, \\ j_{2n+1}^{(as)}(kr) &= (-1)^n j_1^{(as)}(kr) = (-1)^{n+1} \frac{\cos kr}{kr}. \end{aligned} \tag{30}$$

where n is a natural number.

The wave functions $\psi_i(r)$ and $\psi_f(r)$ of the initial i- and the final f-states are linear combinations of the Coulomb functions $F_l(\eta,\rho)$ and $G_l(\eta,\rho)$ (divided by $\rho_{i,f}$, with quantum number $l = 0$ or $l = 1$ for the initial i- or the final f-state, respectively). One can write the Coulomb functions for l in the asymptotic region in this way:

$$F_l(\eta,\rho) = \sin\theta_l, \qquad G_l(\eta,\rho) = \cos\theta_l, \tag{31}$$

where

$$\begin{aligned} \theta_l &= \rho - \eta\log 2\rho + \frac{1}{2}\pi l + \sigma_l(\eta), \qquad \rho_{i,f} = k_{i,f} r, \\ \sigma_l(\eta) &= \arg\Gamma(i\eta + l + 1), \qquad \eta_{i,f} = \frac{m\nu}{k_{i,f}}, \end{aligned} \tag{32}$$

where $\Gamma(x)$ is gamma-function with argument x, ν is Zomerfield parameter.

Now one can conclude that:

- The spherical Bessel function $j_l^{(as)}(kr)$ in the asymptotic region damps (and oscillates) with increasing r equally for any order l.
- The Coulomb functions $F_0(\eta_i,\rho_i)$ and $G_0(\eta_i,\rho_i)$ of order 0 for the initial i-state and the Coulomb functions $F_1(\eta_f,\rho_f)$ and $G_1(\eta_f,\rho_f)$ of order $l = 1$ for the final f-state damp equally in the asymptotic region with increasing r, oscillate equally and are shifted by one phase between each other.
- The total sub-integral function of the integral (19) in the asymptotic region equally damps with increasing r for any order l.

Taking into account (19) and (30), we obtain

$$\begin{aligned} J_c^{(as)}(2n,w) &= (-1)^n J_c^{(as)}(0,w), \\ J_c^{(as)}(2n+1,w) &= (-1)^n J_c^{(as)}(1,w). \end{aligned} \tag{33}$$

I.e. one can reduce any integral inside the asymptotic region to one of two integrals $J^{(as)}(0,w)$ and $J^{(as)}(1,w)$.

Let us find the matrix element $p_c^{(as)}(w,\beta)$ in the asymptotic region:

$$\begin{aligned} p_c^{(as)}(w,\beta) = i\sqrt{\frac{2}{3}} J_c^{(as)}(0,w)\left(\sum_{n=0}^{+\infty}(4n+1)P_{2n}(\cos\beta)\right) \\ + \sqrt{\frac{2}{3}} J_c^{(as)}(1,w)\left(\sum_{n=0}^{+\infty}(4n+3)P_{2n+1}(\cos\beta)\right). \end{aligned} \tag{34}$$

Thus, we reduce formula (11) for the Br spectra to a linear combination of two radial integrals, which are convergent (one can calculate them with a desirable accuracy limited by the effective calculations accuracy of the computer) and do not depend on the angle, and factors-sums on n, by which the problem of convergence is carried out (one can meet with them in (11)).

It would seem that one can cut off the region of numerical integration in one boundary R for the calculation of the integral $J_c(l,w)$ from (19) for any l. However, the calculation convergence of the integral is determined not only by the damping of the sub-integral function at large r, but also by its behavior on the whole integration region. An analysis has shown that the sub-integral function inside the barrier region and inside the external region closer to the barrier behaves so that the calculation of the total integral becomes more and more sensible to it with increasing l and the calculation convergence becomes worse. Therefore, *for obtaining reliable values of the integrals $J_c(l,w)$ (with the same accuracy) it is necessary to increase the external boundary R of the integration region for larger l* (such a conclusion has been obtained by us in calculating the angular Br spectra for ^{210}Po also).

9 Angular calculations for the Br spectra in the α-decay of ^{210}Po

As a demonstration of the method described above, let us calculate the angular Br spectra in the α-decay of ^{210}Po. For a comparison of the results obtained in such approach,

Table 1. Angular values of the Br probability in the α-decay of ^{210}Po in the approximation at $l = 1$.

w, keV	$W_{l=1}(w,\beta)$, 1 / keV / decay							ΔW_1, 1/keV/dec.	ΔW_2
	$\beta = 0°$	$\beta = 15°$	$\beta = 30°$	$\beta = 45°$	$\beta = 60°$	$\beta = 75°$	$\beta = 90°$		
50	1.641E-08	1.635E-08	1.617E-08	1.589E-08	1.553E-08	1.511E-08	1.467E-08	1.735E-09	11.8
100	4.974E-09	4.953E-09	4.892E-09	4.796E-09	4.673E-09	4.531E-09	4.381E-09	5.928E-10	13.5
150	1.897E-09	1.890E-09	1.869E-09	1.836E-09	1.793E-09	1.744E-09	1.692E-09	2.047E-10	12.1
200	8.021E-10	7.993E-10	7.912E-10	7.783E-10	7.618E-10	7.427E-10	7.226E-10	7.949E-11	11.0
250	3.548E-10	3.534E-10	3.493E-10	3.429E-10	3.346E-10	3.250E-10	3.149E-10	3.993E-11	12.7
300	1.611E-10	1.605E-10	1.585E-10	1.554E-10	1.515E-10	1.469E-10	1.421E-10	1.901E-11	13.4
350	7.628E-11	7.590E-11	7.480E-11	7.306E-11	7.082E-11	6.826E-11	6.557E-11	1.071E-11	16.3
400	3.251E-11	3.236E-11	3.194E-11	3.127E-11	3.042E-11	2.943E-11	2.840E-11	4.110E-12	14.5
450	1.278E-11	1.275E-11	1.266E-11	1.252E-11	1.234E-11	1.213E-11	1.191E-11	8.716E-13	7.3
500	6.094E-12	6.077E-12	6.030E-12	5.956E-12	5.860E-12	5.749E-12	5.632E-12	4.615E-13	8.2
550	3.198E-12	3.179E-12	3.124E-12	3.038E-12	2.928E-12	2.802E-12	2.671E-12	5.271E-13	19.7
600	1.624E-12	1.612E-12	1.578E-12	1.524E-12	1.455E-12	1.378E-12	1.297E-12	3.266E-13	25.2
650	5.731E-13	5.712E-13	5.656E-13	5.569E-13	5.456E-13	5.326E-13	5.189E-13	5.422E-14	10.5
700	2.198E-13	2.186E-13	2.150E-13	2.094E-13	2.023E-13	1.942E-13	1.858E-13	3.398E-14	18.3
750	9.515E-14	9.445E-14	9.240E-14	8.920E-14	8.512E-14	8.050E-14	7.571E-14	1.944E-14	25.7
800	2.409E-14	2.411E-14	2.418E-14	2.430E-14	2.446E-14	2.467E-14	2.490E-14	$-8.094E$-16	3.3

with results obtained by models [7,9,8], we shall choose the potential parameters as in [21] (they coincide with the parameters of the potential with the external Coulomb field in [9] and in [7]).

In spite of the fact that there are methods that allow to calculate absolute values of the Br spectra, in this paper at first we shall find the relative values of the Br spectrum for the given nucleus and then we shall normalize the obtained spectra at one selected point of the Br experimental spectrum for the given angle value. This approach in comparison to the previous one allows to analyze a behavior of the Br spectra in dependence on the angle with a larger accuracy (besides being easier to apply).

In the beginning we calculate the total Br probability in the second approximation at $l = 1$ for the 90° angle taking into account (24) (because the component of the Br probability (26) in the second approximation at $l = 1$ equals zero at such angle). Then we normalize the obtained spectrum by the third point of the experimental data [1] (we have values such as $w = 0.179$ keV and $W = 10.1 \cdot 10^{-10}$ 1 / keV / decay), which were also obtained for the angle $\beta = 90°$. Knowing the normalized factor and using formulas (27), we find the Br probability in the second approximation for the other values of the angle β.

The angular values of the Br probability in the second approximation at $l = 1$ are shown in table 1. Here, one can see a variation of the Br probability as a function of the angle β, however this change is extremely small. The Br probability in the first approximation at $l = 0$ coincides with the Br probability in the second approximation at $l = 1$ for the 90° angle. One can see that the contribution of the Br probability in the first approximation to the total spectrum is the largest for any angle value, *i.e.* it is extremely larger than the contribution to the total spectrum of the component of the Br probability in the second approximation at $l = 1$. *This conclusion has a physical sense (obtained for the first time): the Br in the α-decay for ^{210}Po depends extremely weakly on the value*

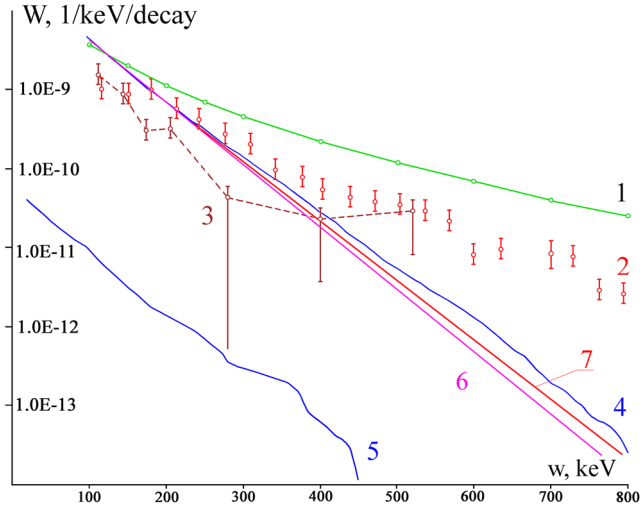

Fig. 1. The Br spectra in the α-decay of ^{210}Po: 1 is the curve, extracted from [9] by the instant accelerated model; 2 is the experimental data [1,4]; 3 is the experimental data [3]; 4 is the curve of the Br probability $W_{l=1}$ in the first approximation at $l = 1$ by our approach at $\beta = 90°$; 5 is the curve of the Br probability component $W^{(l=1)}$ in the first approximation at $l = 1$ at the angle 45° by our approach; 6 is the curve calculated by us with the radial integral (6) and the formula $\mathrm{d}\psi_i(r)/\mathrm{d}r = -\psi_i(r)/w\ \mathrm{d}V(r)/\mathrm{d}r$ (it coincides with the radial integral (7) in [9] with a factor $-1/w$) and further normalization at the third point of data [1,4]; 7 is the curve calculated on the basis of the potential (39)–(41).

of the angle between the directions of the α-particle propagation (or tunneling) and the photon emission (in the given approach). The account of the non-zero component of the Br probability in the second approximation (for angle values which are different from 90°) increases the total Br probability. Absolute and relative variations of the Br probability relatively to its maximal and minimal values,

$$\Delta W_1(w) = W_{l=1}(w, \beta = 0^\circ) - W_{l=1}(w, \beta = 90^\circ),$$

$$\Delta W_2(w) = \frac{|W_{l=1}(w, \beta = 0^\circ) - W_{l=1}(w, \beta = 90^\circ)|}{W_{l=1}(w, \beta = 90^\circ)} \cdot 100, \tag{35}$$

are also included into the table.

The results of the calculations of the Br probability in the α-decay of ^{210}Po in the second approximation at $l = 1$ by our approach are shown in fig. 1.

From the figure one can see, that for the 90° angle our method gives a Br spectrum, which is very close to the Br spectra obtained by the models [7] and [9]. However, with respect to the models [7,9] our method shows an angular variation of the Br spectra.

10 An analysis of the maxima and minima in the angular Br spectra

Let us find the values of the angle β between the directions of the α-particle propagation (or tunneling) and the photon emission, at which the Br probability has the maximal and minimal values. Using the derivative

$$\frac{\mathrm{d}W_{l=1}(w,\beta)}{\mathrm{d}\beta} = W_{l=0}(w)\Big(N^*(w) + N(w) - 2|N(w)|^2\cos\beta\Big)\sin\beta, \tag{36}$$

we find the conditions of extremal values of the function $W_{l=1}(w, \beta)$:

$$\sin(\beta) = 0,$$

$$\cos(\beta) = \frac{N^*(w) + N(w)}{2|N(w)|^2} = \frac{Re(N(w))}{Re(N(w))^2 + Im(N(w))^2}. \tag{37}$$

Calculations for ^{210}Po for the given potential have shown that the second condition in (37) in the range $w = 50$–$800\,\mathrm{keV}$ is not fulfilled. One can explain this by the fact that the integral $J_c(1, w)$ is smaller than the integral $J_c(0, w)$ by 10^2–10^4 times (that is in agreement with the condition of the convergence of the Br spectra with increasing l). From the first condition in (37) we obtain these extremal values for the angle β:

$$\beta = 0, \pi. \tag{38}$$

The Br probability at such angle values has respectively the maximal and minimal values, and for angle values inside that range it varies monotonously for any energy of the photon emitted in the range $w = 50$–$800\,\mathrm{keV}$. One can see this also from table 1.

11 Inclusion of Woods-Saxon potential into the model

Now let us analyze how the Br spectrum in the α-decay of the studied nucleus ^{210}Po changes, if in the approach for calculations of the Br spectra from the interaction potential between the α-particle and the daughter nucleus —pointed out in sect. 9 and having a simplified barrier— one passes to a potential with a barrier, constructed on the basis of the consideration of the realistic nuclear forces of interaction between the α-particle and the daughter nucleus, which is used in realistic nuclear models.

To such a purpose, we shall take the potential proposed in [25] for the description of the α-decay and the synthesis of nuclei. Among the extensive set of literature giving us different types of the α-nucleus potentials, we have given preference to such a paper because there we see a universal and clear approach for the calculation of parameters of the potential after the choice of the desired nucleus. As a result, we suppose to obtain a universal recipe for calculation of the Br spectra after choosing the nucleus.

So, according to [25] (see (6)–(10)), we use this interaction potential:

$$V(r, \theta, l, Q) = V_C(r, \theta) + V_N(r, \theta, Q) + V_l(r), \tag{39}$$

where

$$\begin{aligned} V_C(r,\theta) &= \frac{2Ze^2}{r}\left(1 + \frac{3R^2}{5r^2}\beta Y_{20}(\theta)\right), \\ V_N(r,\theta,Q) &= \frac{v(A,Z,Q)}{1 + \exp\dfrac{r - r_0}{d}}, \\ V_l(r) &= \frac{l(l+1)}{2mr^2}. \end{aligned} \tag{40}$$

At the current stage (with the purpose to simplify the numerical calculations of the Br spectra), for the determination of the component $V_C(r, \theta)$ we use formula (7) of [25] on the whole region of r (without using (8) of [25]).

According to (14)–(20) of [25], we calculate the parameters as follows:

$$\begin{aligned} v(A,Z,Q) &= -(30.275 - 0.45838Z/A^{1/3} + 58.270I \\ &\quad -0.24244Q), \\ r_m &= 1.5268 + R, \\ R &= R_p(1 + 3.0909/R_p^2) + 0.1243t, \\ R_p &= 1.24A^{1/3}(1 + 1.646/A - 0.191I), \\ t &= I - 0.4A/(A + 200), \\ d &= 0.49290. \end{aligned} \tag{41}$$

According to [26], we see that the parameter β for ^{210}Po is very small, and this points out the high degree of sphericity of this nucleus. Therefore, for the calculation of the Br spectra we note the following:

– In the definition of $r_m(\theta)$ and $R(\theta)$ we do not use (21) and (22) of [25] (use is made of (15) from [25]).
– The formalism for the calculation of the Br spectra (presented in sect. 3–10), is constructed on the basis of the division of the total wave function into its radial and angular components, *i.e.* under the assumption of spherical symmetry of the decaying nucleus. *Therefore, for the nucleus ^{210}Po our approach for calculation of the Br spectra is applicable taking into account realistic nuclear forces too.*

Moreover, we calculate the radial wave function of the decaying system for the potential (39)–(41). This gives us the general solution for the wave function as a function of the selected energy level for the α-decay. To obtain that the found solutions describe the states of the decaying system before and after the spontaneous photon emission, we should take into account boundary conditions in initial and final states. Here, we use the following conditions:

$$\begin{aligned} &\text{initial } i\text{-state: } \chi_i(r \to +\infty) \to G(r) + iF(r), \\ &\text{final } f\text{-state: } \chi_f(r=0) = 0, \end{aligned} \tag{42}$$

where $\varphi_{i,f}(r) = \dfrac{\chi_{i,f}(r)}{r}$, F and G are Coulomb functions.

One should note that in contradiction to a scattering of the α-particle on the nucleus, where as boundary condition for the initial i-state *a finiteness of the radial wave function $\varphi_i(r)$ should be used at point $r = 0$ ($\chi_i(r = 0) = 0$)*, for the decay we choose the natural requirement that the radial wave function tends to a form of the radial spherical wave divergent outside in the asymptotic region ($\chi_i(r \to \infty)$ tend to a plane wave moving to the right). *This condition gives us inevitably the divergence of the total radial wave function in the initial i-state at point $r = 0$ (whose real and imaginary parts consist of regular and singular solutions)!* One can make sure about this by requiring the fulfillment of the continuity condition for the radial wave function on the whole region by its definition; or by requiring the constancy of the radial flux density, which is distinct from zero and is directed outside in the asymptotic region (and, therefore, it should be non-zero near the point $r = 0$, that is impossible to execute with null wave function at any chosen point of r). This peculiarity essentially complicates the calculations of the Br spectra for the α-decay in comparison with the problems of the calculation of the Br spectra for the scattering of charged particles on nuclei (where considerable progress has been achieved and a lot of papers have been published).

It turns out that the real and imaginary parts of the sub-integral function of (17) for the calculation of the matrix elements, constructed on the basis of the found solutions for the wave function for the initial and final states, tend to zero at the point $r \to 0$! This interesting peculiarity provides the convergence of the matrix elements near the point $r = 0$ and, therefore, in the whole region of r (in the asymptotic region the convergence of the wave function is determined by the convergence of the Coulomb functions, considered above). Thus, we resolve the divergence problem in the calculations of the Br spectra in the α-decay of the nucleus ^{210}Po.

The Br spectrum for the nucleus ^{210}Po with the α-nucleus potential (39)–(41) by our approach and calculations is shown by the curve labelled with number 7 in fig. 1. From here one can see that the new curve 7 of the Br probability is located very close to the curve 4 for the Br probability with the potential from sect. 9 with a simplified barrier (and also close to the curve 6 from approaches [7,9]).

Conclusions:

– The consideration of nuclear forces essentially changed the shape of the barrier in its internal part (and essentially changed the Br from this internal region), and changed very slowly the spectrum of the total Br in the α-decay of the nucleus ^{210}Po (in comparison with the earlier obtained Br spectrum on the basis of the potential from sect. 9 with a simplified barrier).
– This point confirms the result (obtained earlier on the basis of the α-nucleus potential pointed out in sect. 9 with a barrier of simplified shape) that the Br emission from the internal region up to the point r for the barrier maximum gives a very small contribution to the total Br spectrum. This conclusion coincides logically with the property (found on the basis of microscopic models of nuclei with their α-decay) of propagation of the α-particle from the nuclear surface during the first decay stage.

12 Conclusions and perspectives

We presented a new method of calculation of BrS spectra in the α-decay, where the angle between the directions of the α-particle motion (with tunneling) and the photon emission (used in the experiments [4,5]) is taken into account. Using it, the angular spectra for the nucleus ^{210}Po are obtained for the first time. Now let us formulate the main conclusions and perspectives:

– The method gives a dependence of the bremsstrahlung spectrum in the α-decay of ^{210}Po on the angle (this has been obtained for the first time):
  – the first approximation at $l = 0$ gives independence of the spectrum from the angle;
  – the second approximation at $l = 1$ gives a slow monotonous variation of the slope of the spectrum curve with changing the angle and without a visible change of the shape of the spectrum curve (*i.e.* without the appearance of humps and holes in the spectrum);
  – for arbitrary energy of the photon emitted in the range of $w = 50$–750 keV the bremsstrahlung probability is maximal at the angle 0° and is minimal at the angle 180°, between these angular values the bremsstrahlung probability varies monotonously.
– Results for ^{210}Po have obtained on the basis of these approximations:

  - the bremsstrahlung process does not depend on the direction of the leaving α-particle relatively to the shape of the daughter nucleus before the photon emission (this assumption has been fulfilled for ^{210}Po, because, in accordance with [26] (see fig. 5 on p. 33), coefficients β^0_λ of the shape deformation for this nucleus at $\lambda = 2, 4, 6, 8$ are extremely close to zero in comparison with other nuclei with other numbers of protons and neutrons, *i.e.* ^{210}Po is one of the most spherical nuclei);
  - the photon emission does not change the direction of the α-particle propagation (this assumption is suitable for the low-energy photons, and, therefore, it is applicable for analysis of all existing experimental data of the bremsstrahlung spectra, where one can select a region with smaller photons energies for increasing the accuracy);
  - the bremsstrahlung spectra have been calculated by means of the α-nucleus potential with the simplified barrier pointed out in sect. 9 (they coincide with the α-nucleus potential in [21], and also with the potential with the external Coulomb field in [9] and in [7]) and with use of the α-nucleus potential with the barrier, pointed out in sect. 11 (see [25]) and constructed on the basis of realistic nuclear forces.
- Taking into account nuclear forces in the method gives the following:
  - it does not change dependences on the angle of the bremsstrahlung probability in the first and second approximations;
  - it essentially changes the shape of the barrier in its internal region (sufficiently changes the Br from such internal region) and changes very little the spectrum of the total Br in the α-decay for the nucleus ^{210}Po at selected angle values.

From here a question naturally arises: *which improvement should be made in the method that results in enough visible changes of the Br spectrum curve, to achieve a better description of the experimental data?* Note the following:

- According to our calculations, in consideration of the possibility of the α-particle leaving at the energy of the *excited state* of the decaying system, the angle of the slope of the Br spectrum curve increases (monotonously). Apparently, it allows to displace the calculated Br curve (for example, by our method) essentially closer to the experimental data [1].
- For obtaining *reliable values* of the Br spectra for the α-decay one needs to achieve in the calculations the convergence of integrals for such spectra. This leads to the necessity to consider wave functions inside a large space region with the external boundary far enough from the nucleus. From here a new question naturally arises as to taking into account electrons shells of the atom with such nuclear α-decay in the calculation of the Br spectra in the α-decay and one can formulate the following *hypothesis about the visible influence of the electrons shells of the atom on the total Br spectrum in the α-decay.* One can note that an essential progress has been made by M. Amusia in the study of the Br in atomic physics [27,28]. Moreover, according to [29] (see pp. 20–21), there is an inevitable influence of the α-decay process at its starting time stage on the electrons shells of the atom whose nucleus decays. So, the α-particle during its propagation (with tunneling) deforms and polarizes these electrons shells. In one's turn, the changed electrons shells can correct our understanding of the real α-nucleus potential in the model, which should be used for the calculation of the Br spectra (one should note that these effects are still not studied in details). Note that these effects (partially) take place in the same space region, where it is necessary to use the wave functions for the calculation of the matrix elements to achieve converging values of the Br spectra. Therefore, it is desirable to study these effects to obtain more accurately the total Br spectra of the α-decay.
- The inclusion of the α-nucleus potential from [25] in our method, deforming the decay barrier, does not essentially displace a point, where the α-particle starts to tunnel through the barrier. It turns out that the displacement of this point is much smaller in comparison with the tunneling region and even with a *"mixed region"*, as introduced in [8]. Therefore, after taking into account the realistic nuclear forces in the method, the interest to analyze the Br from these regions (with detailed study of the tunneling) remains.
- We assume that further development of the time formalism for the description of the Br in the α-decay at its first stage will give new abilities in the accurate description of the Br. One cannot exclude the assumption about the appearance of *"holes"* in the Br spectra (see [2,8]), that can allow to better describe the experimental data [2,3]. However, in such a case it is not clear how to connect this with the available experimental data [1] without "holes" in the logical basis of our method.

Now let us formulate conclusions which have a physical sense and on the basis of the calculations for ^{210}Po by our model:

- The bremsstrahlung in the α-decay of spherical nuclei depends on the angle extremely weakly. Taking into account nuclear forces, such dependence does not change visibly.
- It is not enough to take into account only one angle for the explanation of the difference between the experimental spectra [1] and [2,3] for ^{210}Po (which equals 90° and 25°, respectively) on the basis of our model and for the explanation of the difference between these experimental spectra and the calculated curves averaged by angle values in approaches [7–9] (that can be supposed in [4,5]).
- The small visible change of the Br spectra after taking into account the realistic nuclear forces in our method confirms the result (obtained on the basis of the α-nucleus potential from sect. 9 with the barrier of simplified shape, see also [21]) that the Br from the inter-

nal region till point r for the barrier maximum gives a very small contribution to the total Br spectrum. This conclusion becomes natural if we take into account such a property (found on the basis of microscopic models of the α-decay) as the α-decay starts when the α-particle leaves the nuclear surface.

In closing, supposing that the Br spectra in the α-decay must change essentially with changing the angle value, we note how this point can be explained, analyzing this question from the following theoretical and experimental aspects:

- One can explain this angular change of the bremsstrahlung spectrum in this way:
  - In the α-decay of (initially) spherical nuclei —by the strong angular deformation of the decay barrier and continuous redistribution of the electromagnetic charge (or *"nuclear polarization"* like the polarization of the electrons shells during tunneling of the α-particle, according to [29] (see p. 20–21)). One can suppose that here non-central forces between the α-particle and nucleons of the daughter nucleus, which exist in the barrier region mainly, play an essential role. Note that a serious progress was achieved early in a microscopic approach to study the bremsstrahlung in scattering of the nucleons and of the α-particles on light nuclei (see [30–34]), in a study of the bremsstrahlung in collisions between heavy ions and the nuclei (see [35]), and in the non-microscopic approaches to study the bremsstrahlung induced by protons during their collisions on heavier nuclei (see [36]).
  - In the α-decay of deformed nuclei —by the essential appearance of the angular anisotropy of the α-nucleus potential. Then one can extract an information about the shape of the nucleus from the angular bremsstrahlung spectra.
- Experimental confirmation of the change of the Br spectrum in the α-decay of the spherical nuclei with changing the angle value gives the following:
  - It will prove the existence of important microscopic forces between the α-particle and the nucleons of the daughter nucleus, reinforcing the angular deformation of the barrier.
  - It will prove a visible influence of the bremsstrahlung on the dynamics of the α-decay. In accordance with our model, it will be an experimental confirmation of the effect of variation of the barrier penetrability as a result of the emission during tunneling of a charged particle, as proposed in [20].

The angular analysis of the bremsstrahlung spectra gives a new additional information about the α-decay. Therefore, further angular experimental measurements of the bremsstrahlung spectra will be of interest.

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